

CS103  
WINTER 2026



# Lecture 07: **Functions**

**Part 2 of 2**

# Functions

## Part 2

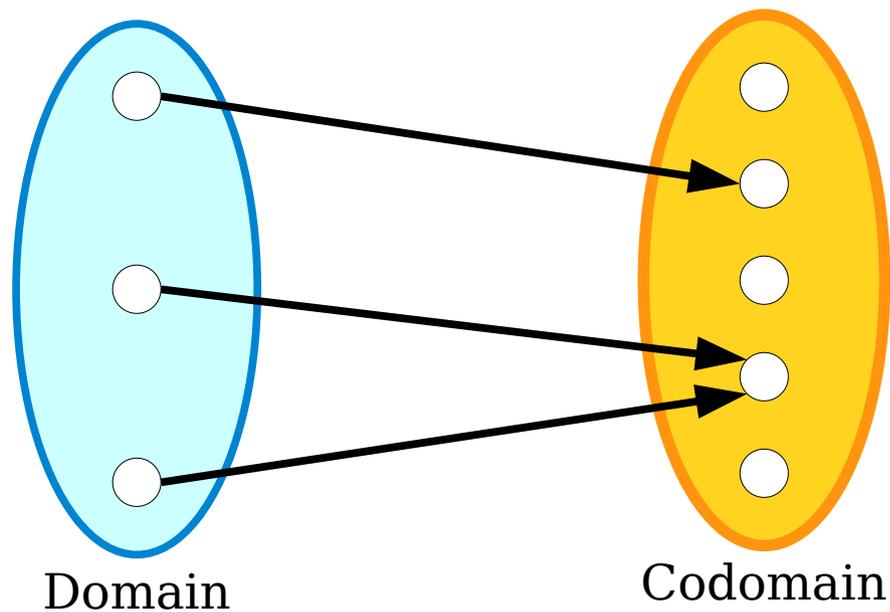
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4. Leveling Up the Assume-Prove Table
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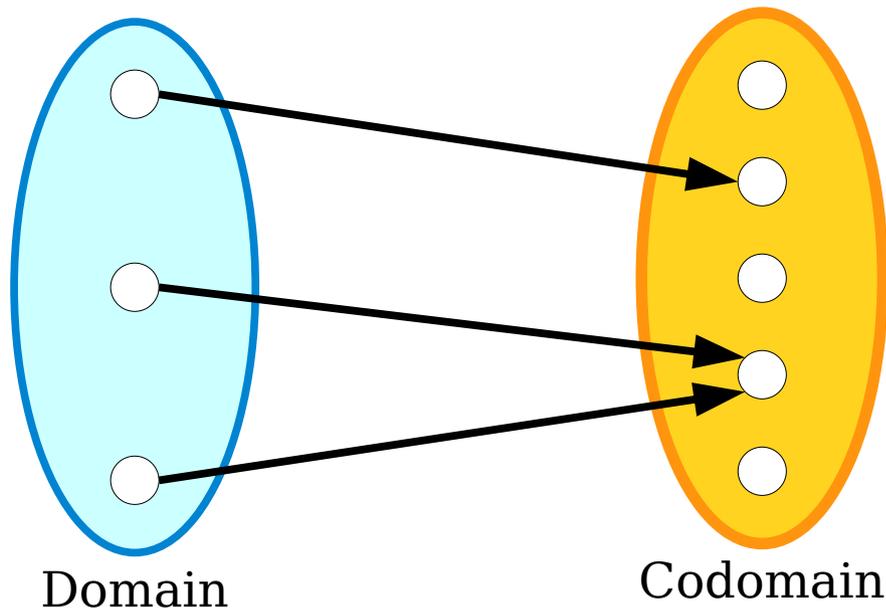
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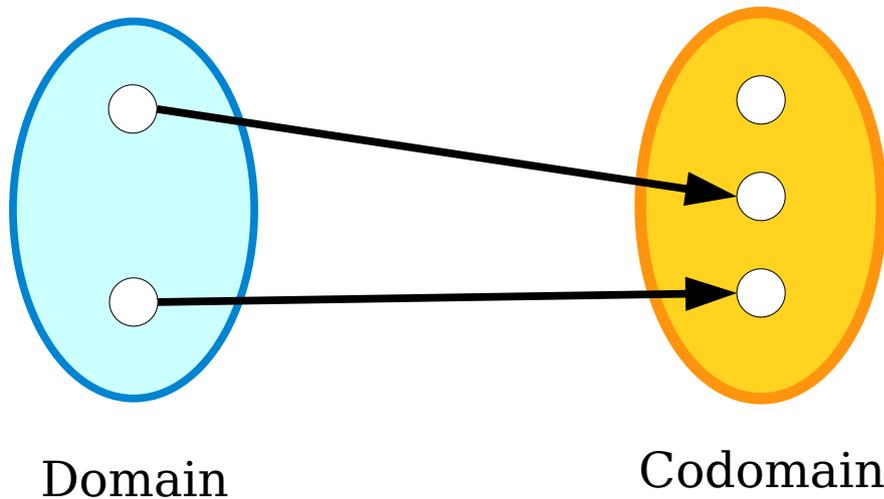


*Is it a **function**?* **Yes!**

*Is it an **injection**?* **No.**

*Is it a **surjection**?* **No.**

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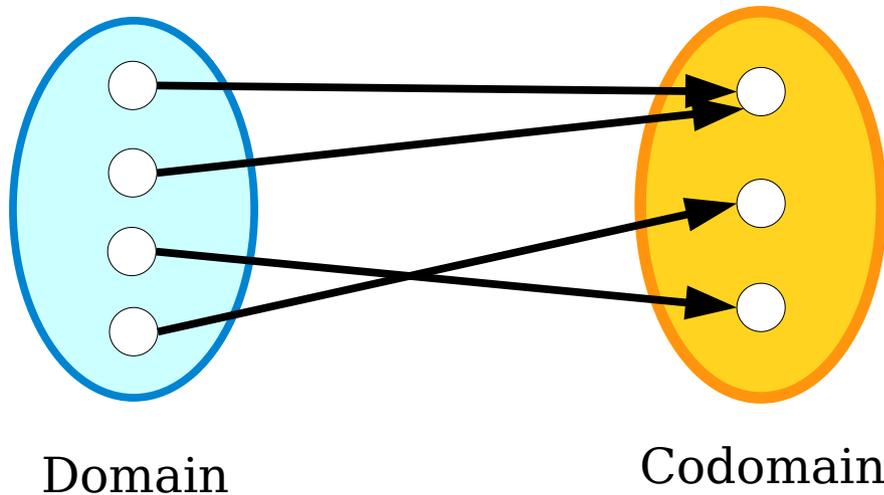


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# Recap from Last Time



*Is it a **function**?*    **Yes!**

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# Recap from Last Time

**Injection:**  $\forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \rightarrow f(a_1) \neq f(a_2))$

*If the inputs are different, the outputs are different*

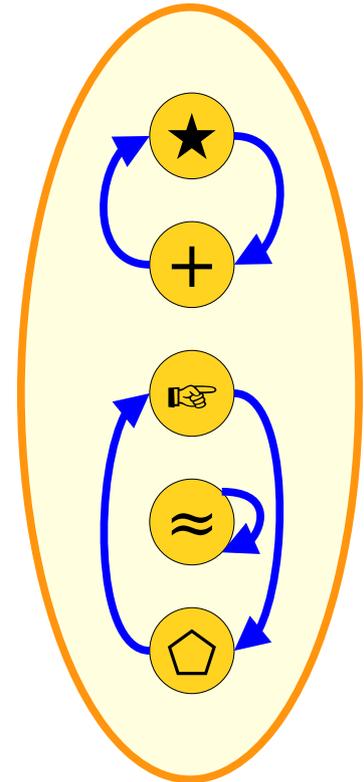
- Can also define with the contrapositive!

**Surjection:**  $\forall b \in B. \exists a \in A. f(a) = b$

*“For every possible output, there's an input that produces it.”*

**Involution:**  $\forall x \in A. f(f(x)) = x$

*“Applying  $f$  twice is equivalent to not applying  $f$  at all.”*



# The Assume-Prove Table

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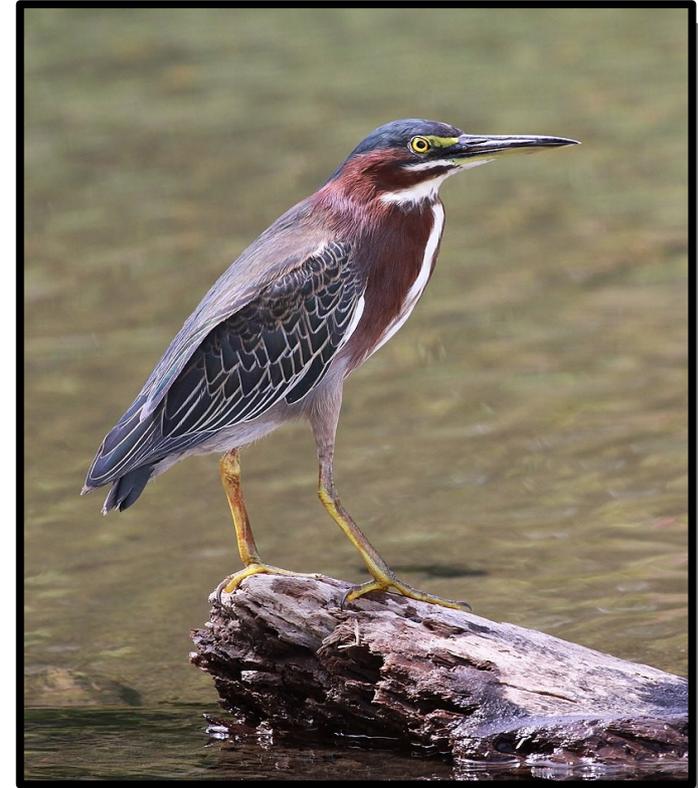
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***Theorem:*** If all birds have feathers,  
then all herons have feathers.

# A Proof About Birds

**Theorem:** If all birds have feathers, then all herons have feathers.

Given the predicates

*Bird*(*b*), which says *b* is a bird;

*Heron*(*h*), which says *h* is a heron; and

*Feathers*(*x*), which says *x* has feathers,

translate the theorem into first-order logic.

Answer at

[cs103.stanford.edu/polle](https://cs103.stanford.edu/polle)

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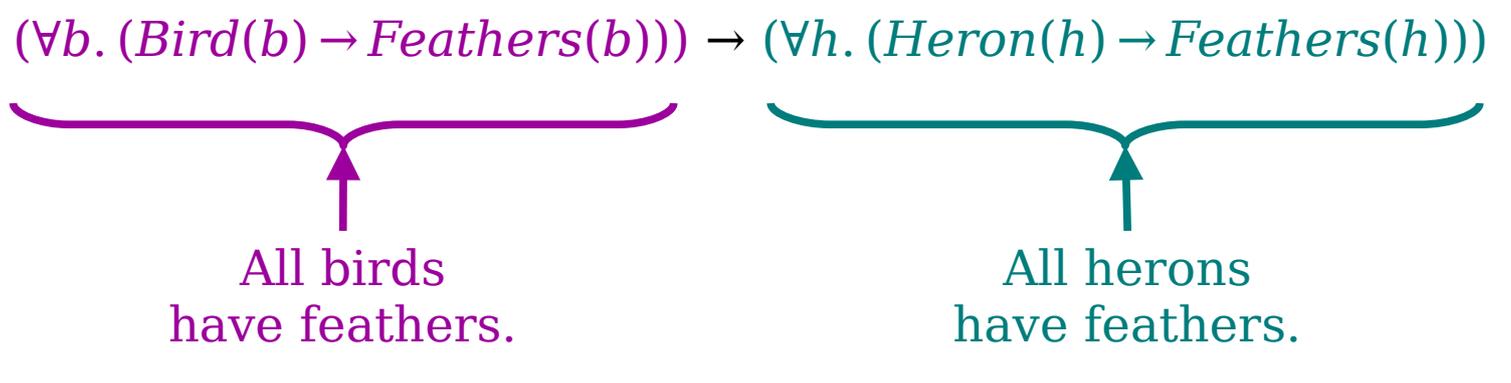
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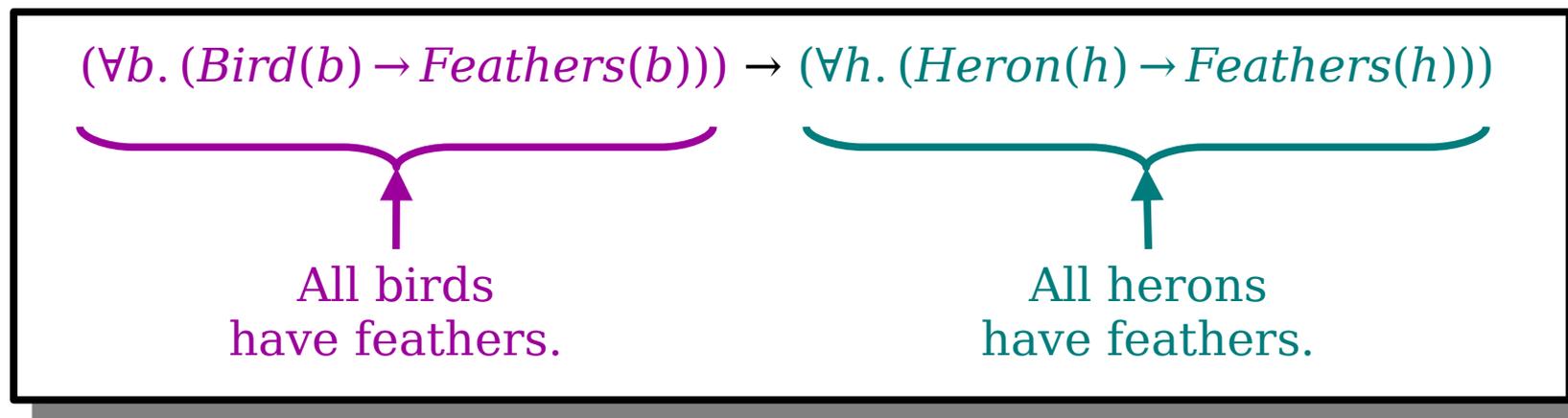
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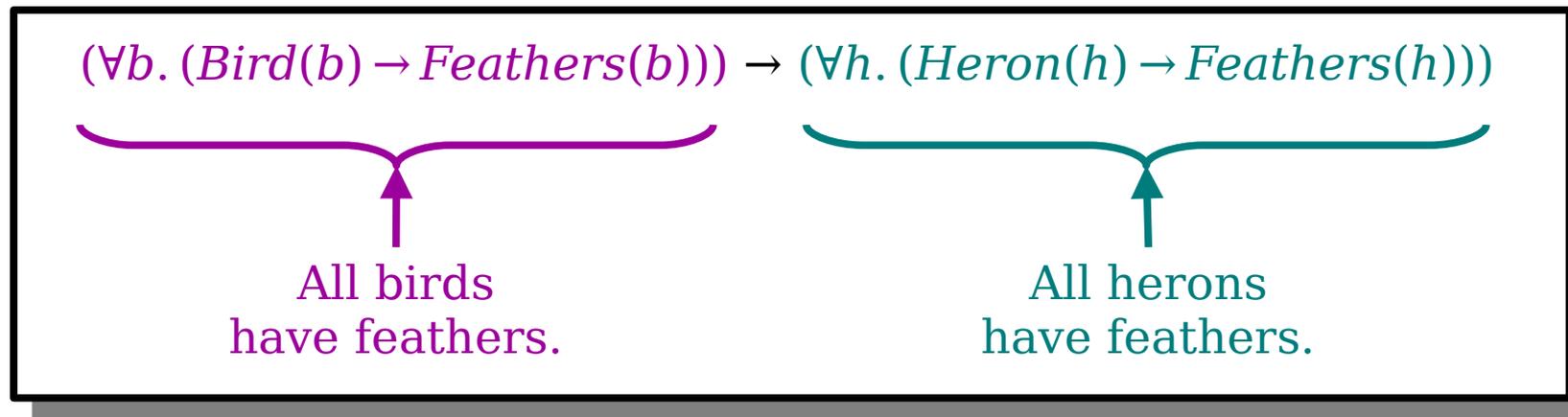


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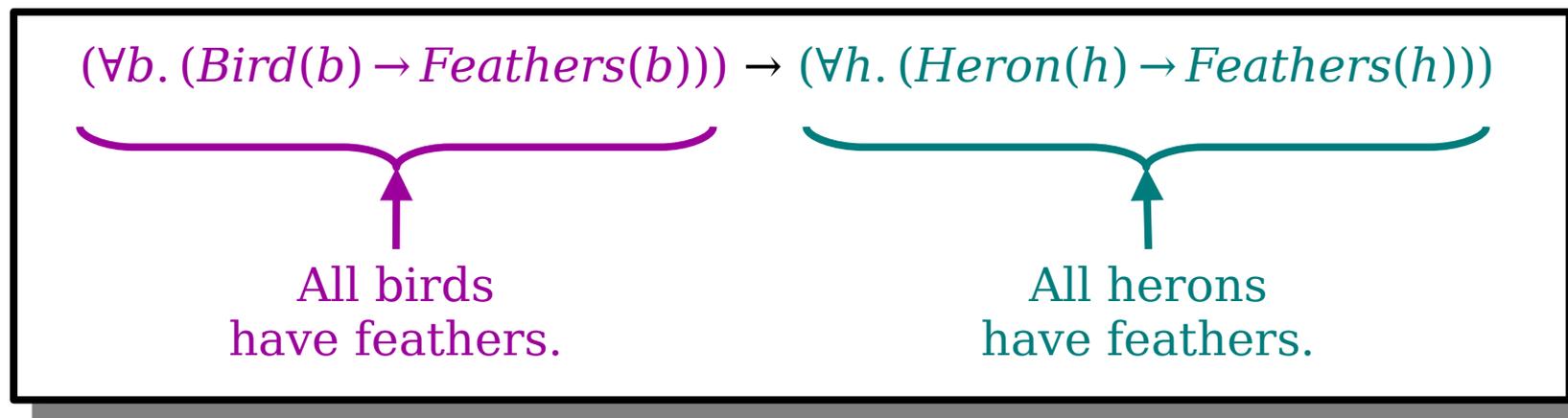


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Consider an arbitrary bird  $b$ . Since  $b$  is a bird,  $b$  has feathers. *[And now we're stuck! We are interested in herons, but  $b$  might not be one. It could be a hummingbird, for example!]*



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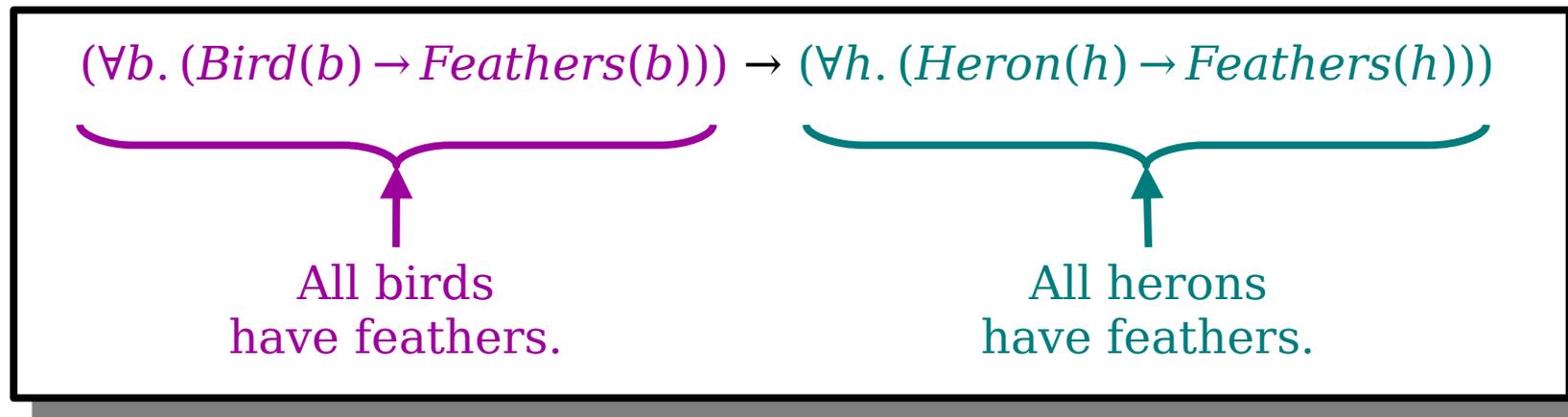
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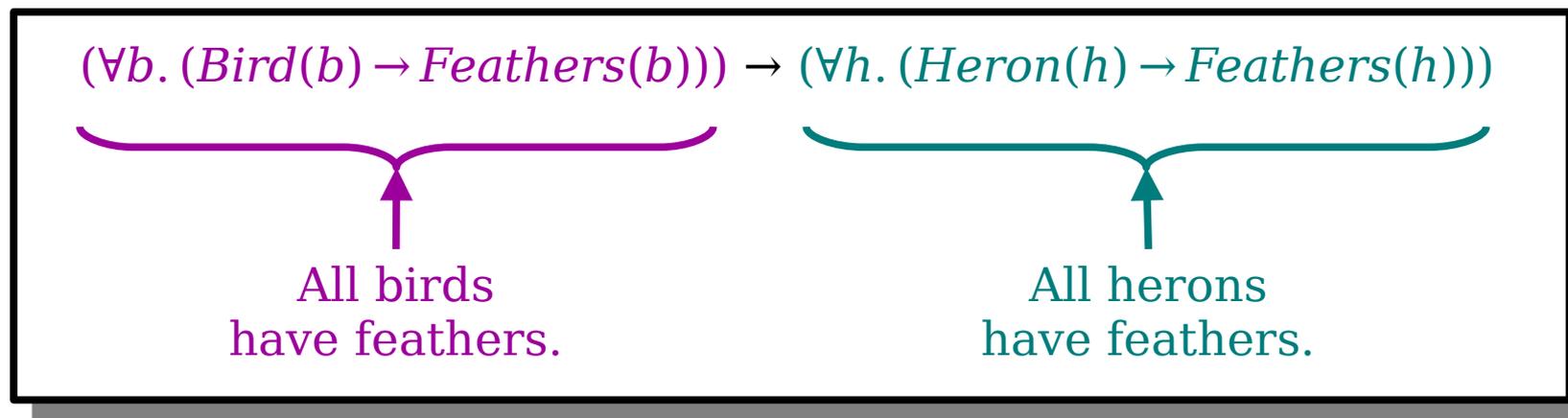


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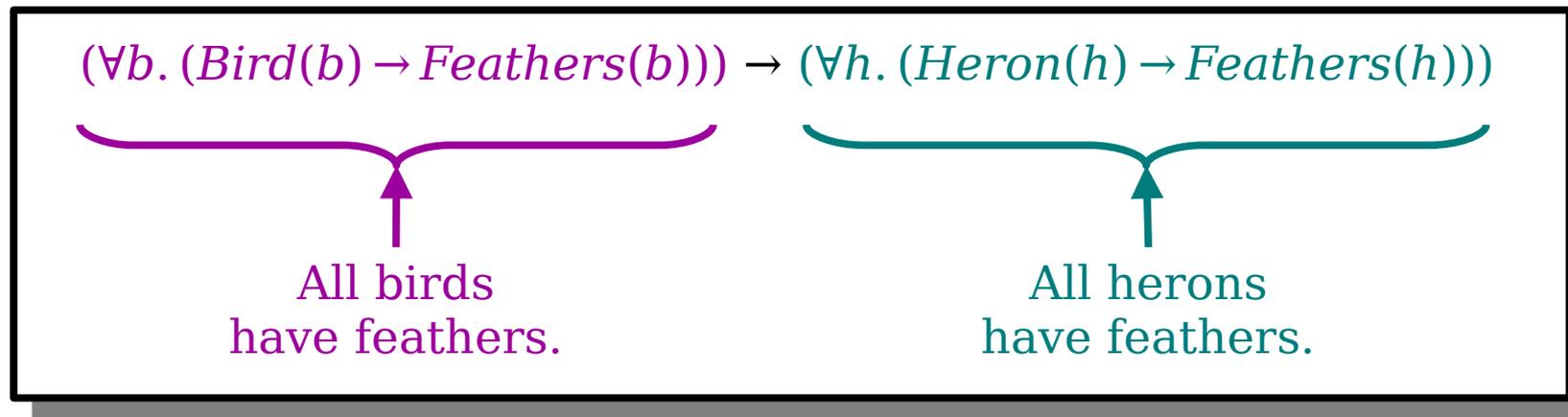


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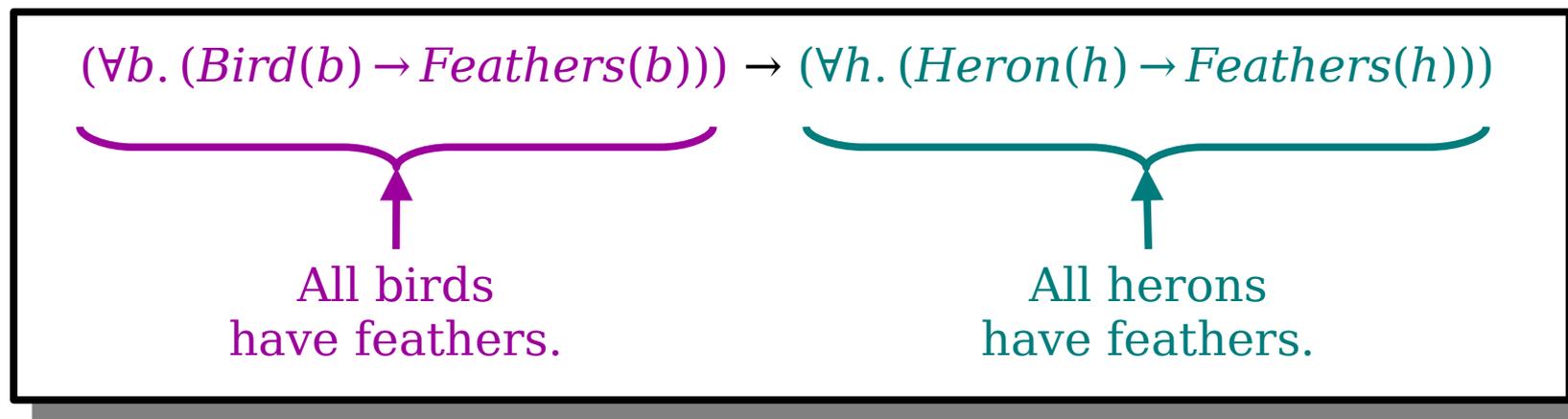


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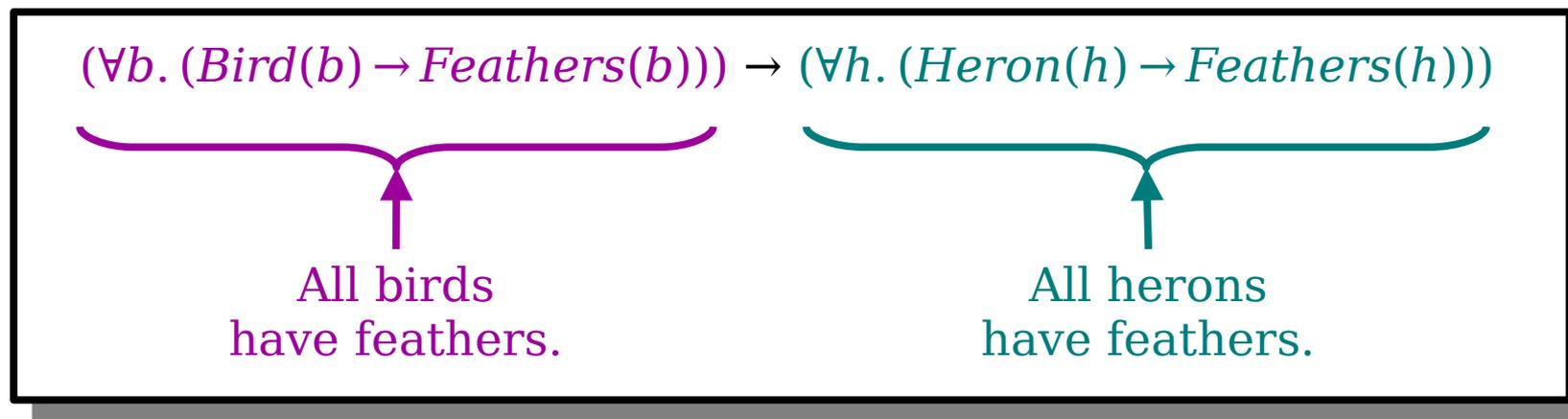


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We never introduce a variable  $b$ .

We introduce a variable  $h$  almost immediately.

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# Proving vs. Assuming

- In the context of a proof, you will need to assume some statements and prove others.
  - Here, we **assumed** all birds have feathers.
  - Here, we **proved** all herons have feathers.
- Statements behave differently based on whether you're assuming or proving them.

$(\forall b. (Bird(b) \rightarrow Feathers(b))) \rightarrow (\forall h. (Heron(h) \rightarrow Feathers(h)))$

We never introduce a variable  $b$ .

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# Proving vs. Assuming

- To **prove** the universally-quantified statement

$$\forall x. P(x)$$

we introduce a new variable  $x$  representing some arbitrarily-chosen value.

- Then, we prove that  $P(x)$  is true for that variable  $x$ .
- That's why we introduced a variable  $h$  in this proof representing a heron.

$$(\forall b. (Bird(b) \rightarrow Feathers(b))) \rightarrow (\forall h. (Heron(h) \rightarrow Feathers(h)))$$

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# Proving vs. Assuming

- If we **assume** the statement

$$\forall x. P(x)$$

we **do not** introduce a variable  $x$ .

- Rather, if we find a relevant value  $z$  somewhere else in the proof, we can conclude that  $P(z)$  is true.
- That's why we didn't introduce a variable  $b$  in our proof, and why we concluded that  $h$ , our heron, have feathers.

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# Types of Functions

- We now have three special types of functions:
  - ***involutions***, functions that undo themselves;
  - ***injections***, functions where different inputs go to different outputs; and
  - ***surjections***, functions that cover their whole codomain.
- ***Question:*** How do these three classes of functions relate to one another?

***Theorem:*** For any function  $f : A \rightarrow A$ , if  $f$  is an involution, then  $f$  is surjective.

$$\underbrace{(\forall x \in A. f(f(x)) = x)}_{\substack{f \text{ is an} \\ \text{involution.}}} \rightarrow \underbrace{(\forall b \in A. \exists a \in A. f(a) = b)}_{\substack{f \text{ is} \\ \text{surjective.}}}$$

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**Theorem:** For any function  $f : A \rightarrow A$ , if  $f$  is an involution, then  $f$  is surjective.

$$(\forall x \in A. f(f(x)) = x) \rightarrow (\forall b \in A. \exists a \in A. f(a) = b)$$

Assume this.

Prove this.

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Assume this.

Prove this.

$$(\forall b. (Bird(b) \rightarrow Feathers(b))) \rightarrow (\forall h. (Heron(h) \rightarrow Feathers(h)))$$

Assume this.

Prove this.

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$$(\forall x \in A. f(f(x)) = x) \rightarrow (\forall b \in A. \exists a \in A. f(a) = b)$$

Assume this.

Prove this.

If you **assume** this is true...

Initially, **do nothing**. Once you find a  $z$  through other means, you can state it has property  $A$ .

**Theorem:** For any function  $f : A \rightarrow A$ , if  $f$  is an involution, then  $f$  is surjective.

$$(\forall x \in A. f(f(x)) = x) \rightarrow (\forall b \in A. \exists a \in A. f(a) = b)$$

Assume this.

Prove this.

Since we're assuming this, we aren't going to pick a specific choice of  $x$  right now. Instead, we're going to keep an eye out for something to apply this fact to.

### ***Proof Outline***

1. Assume  $f$  is an involution.

***Theorem:*** For any function  $f : A \rightarrow A$ , if  $f$  is an involution, then  $f$  is surjective.

$$(\forall x \in A. f(f(x)) = x) \rightarrow$$

$$(\forall b \in A. \exists a \in A. f(a) = b)$$

We've said that we need to prove this statement. How do we do that?

Prove this.

What do you do to prove  $\forall b \in A. [\text{something}]$ ?

Answer at

<https://cs103.stanford.edu/pollev>

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To **prove** that this is true...

Have the reader pick an arbitrary  $x$ . Then prove  $A$  is true for that choice of  $x$ .

Prove this.

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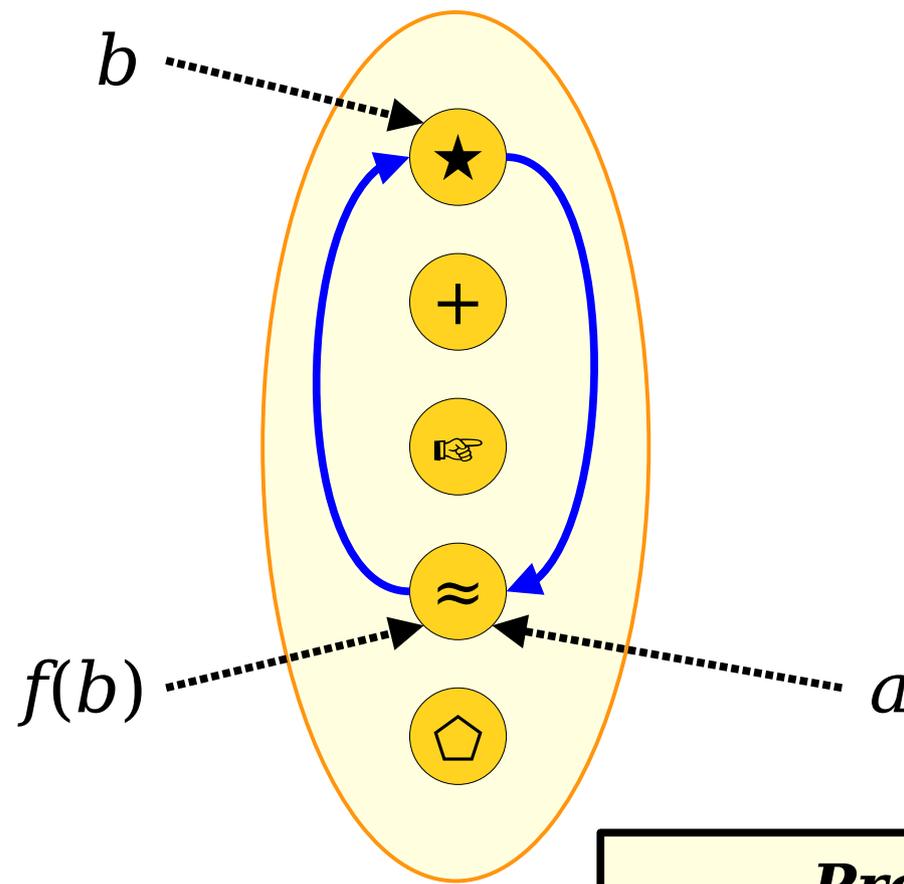
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This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

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# Functions

## Part 2

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***Theorem:*** Let  $f : A \rightarrow A$  be an involution.  
Then  $f$  is injective.

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Then  $f$  is injective.

**What We're Assuming**

$f : A \rightarrow A$  is an involution.

$$\forall z \in A. f(f(z)) = z.$$

We're *assuming* this universally-quantified statement, so we won't introduce a variable for what's here.

**What We Need to Prove**

$f$  is injective.

$$\forall a_1 \in A. \forall a_2 \in A. (f(a_1) = f(a_2) \rightarrow a_1 = a_2)$$

We need to *prove* this universally-quantified statement. So let's introduce arbitrarily-chosen values.

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We need to prove this **implication**. So we **assume the antecedent** and **prove the consequent**.

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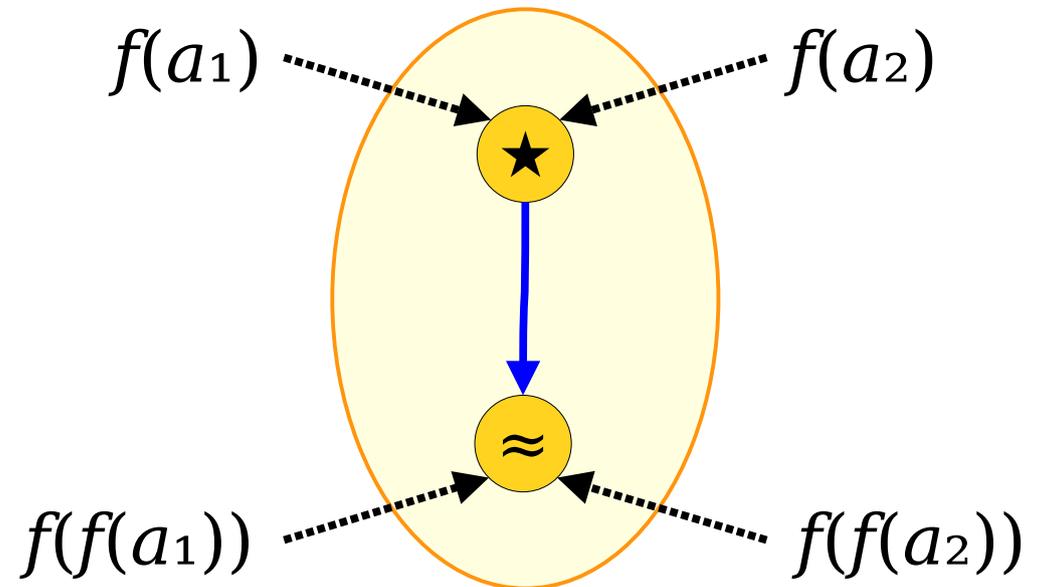
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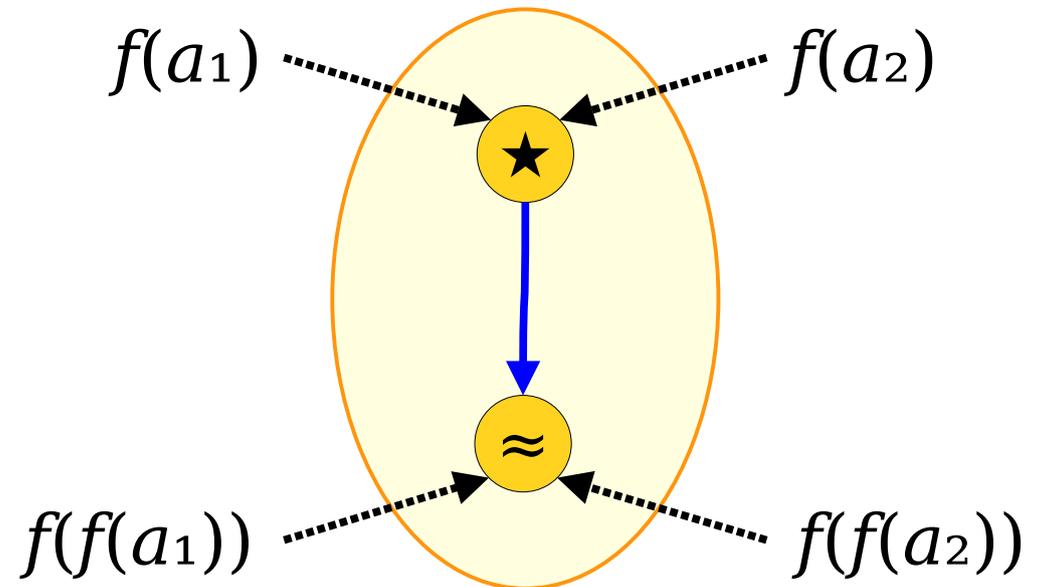
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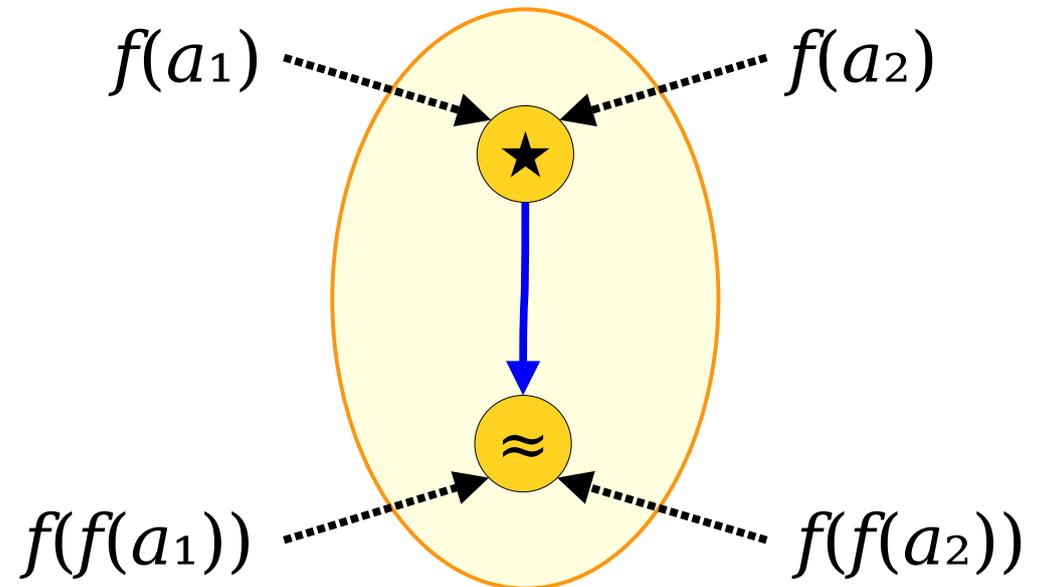
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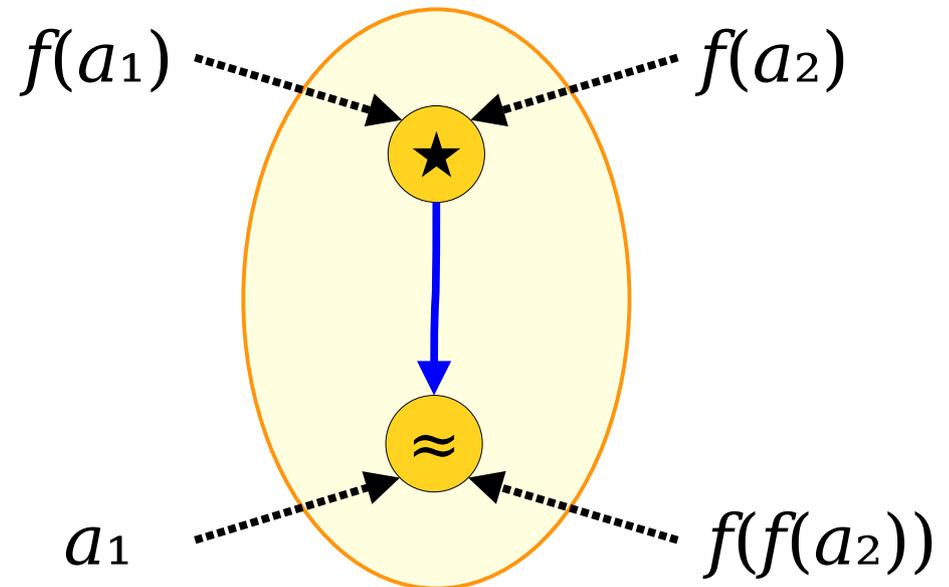
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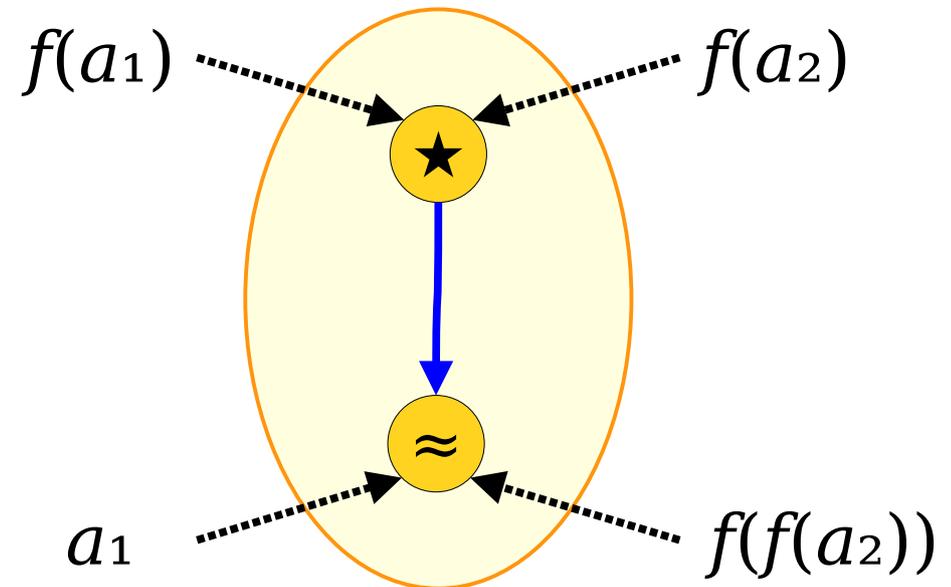
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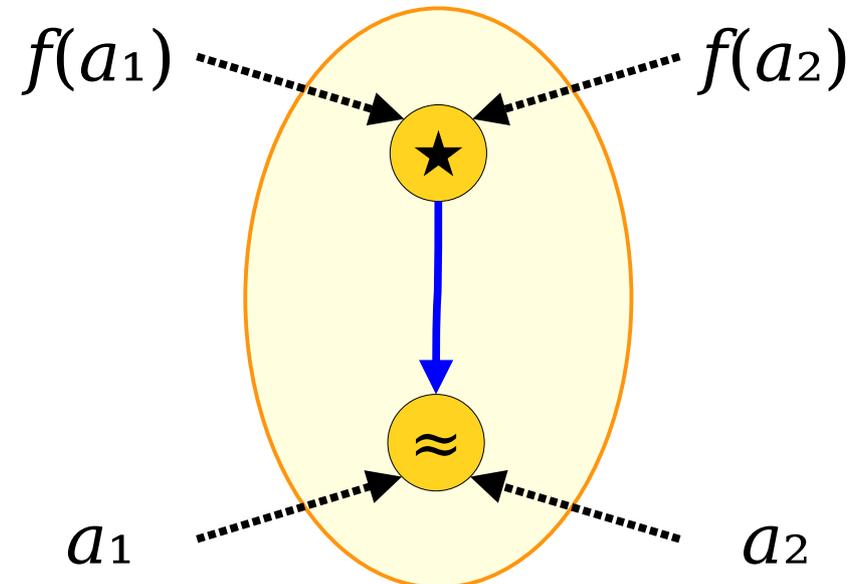
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Since  $f(a_1) = f(a_2)$ , we know that  $f(f(a_1)) = f(f(a_2))$ .

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so  $a_1 = a_2$ , as needed.

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This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

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# Announcements

- ***Problem Set Three***
  - posted today, due next Friday at 1:00 PM
  - some parts require Monday's lecture; clearly labeled
- ***Problem Set Two***
  - can use a late day to submit by Saturday at 1:00 PM
- ***Left-handed desk form***
  - due Monday of next week
- ***Attendance opt-out form***
  - available sometime next Monday, due next Friday

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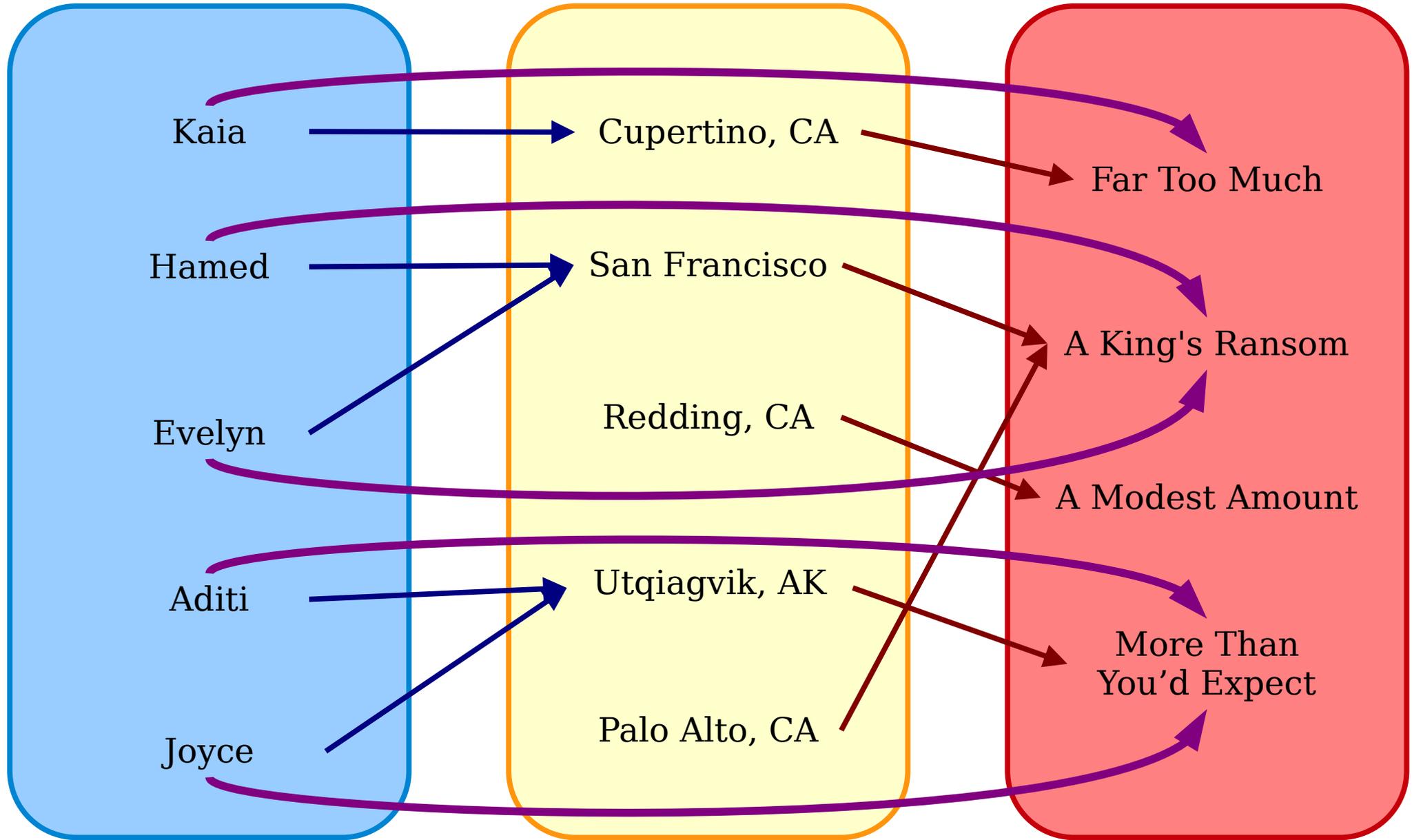
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***f : People → Places***

***g : Places → Prices***



*People*

*Places*

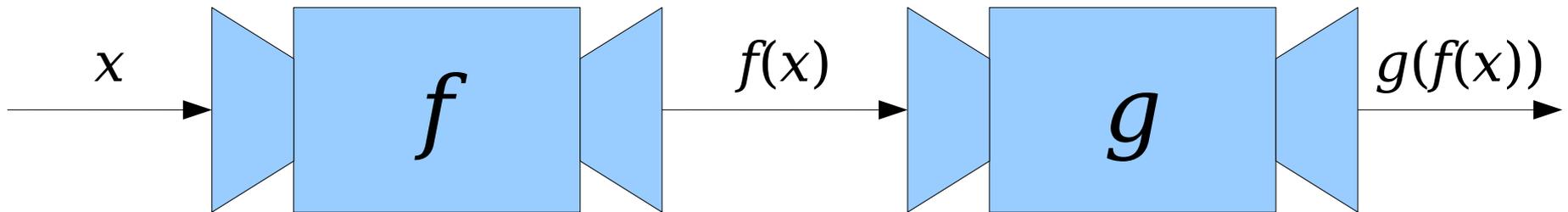
*Prices*

***h : People → Prices***

***$h(x) = g(f(x))$***

# Function Composition

- Suppose that we have two functions  $f : A \rightarrow B$  and  $g : B \rightarrow C$ .
- Notice that the codomain of  $f$  is the domain of  $g$ . This means that we can use outputs from  $f$  as inputs to  $g$ .



# Function Composition

- Suppose that we have two functions  $f : A \rightarrow B$  and  $g : B \rightarrow C$ .
- The **composition of  $f$  and  $g$** , denoted  $g \circ f$ , is a function where
  - $g \circ f : A \rightarrow C$ , and
  - $(g \circ f)(x) = g(f(x))$ .
- A few things to notice:
  - The domain of  $g \circ f$  is the domain of  $f$ . Its codomain is  $g$ 's codomain.
  - Even though composition is written  $g \circ f$ , when evaluating  $(g \circ f)(x)$ , the function  $f$  is evaluated first.
- Composition is **associative**:  $(f \circ g) \circ h = f \circ (g \circ h)$ . (Prove this!)
- Composition is not necessarily commutative:  $f \circ g$  is not necessarily the same as  $g \circ f$ . (Prove this!)

The name of the function is  $g \circ f$ .  
When we apply it to an input  $x$ , we write  $(g \circ f)(x)$ . I don't know why, but that's what we do.

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### What We're Assuming

$f : A \rightarrow B$  is an injection.

$\forall x \in A. \forall y \in A. (x \neq y \rightarrow$   
 $f(x) \neq f(y))$

$g : B \rightarrow C$  is an injection.

$\forall x \in B. \forall y \in B. (x \neq y \rightarrow$   
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We're *assuming* these universally-quantified statements, so we won't introduce any variables for what's here.

### What We Need to Prove

$g \circ f$  is an injection.

$\forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \rightarrow$   
 $(g \circ f)(a_1) \neq (g \circ f)(a_2))$

We need to *prove* this universally-quantified statement. So let's introduce arbitrarily-chosen values.

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Now we're looking at an implication. Let's *assume* the antecedent and *prove* the consequent.

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Let's write this out separately and simplify things a bit.

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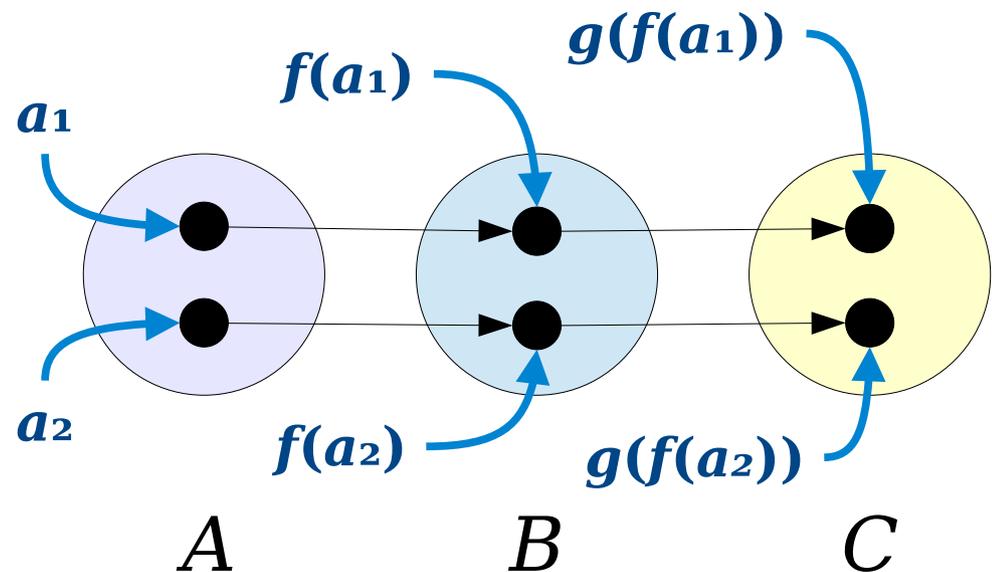
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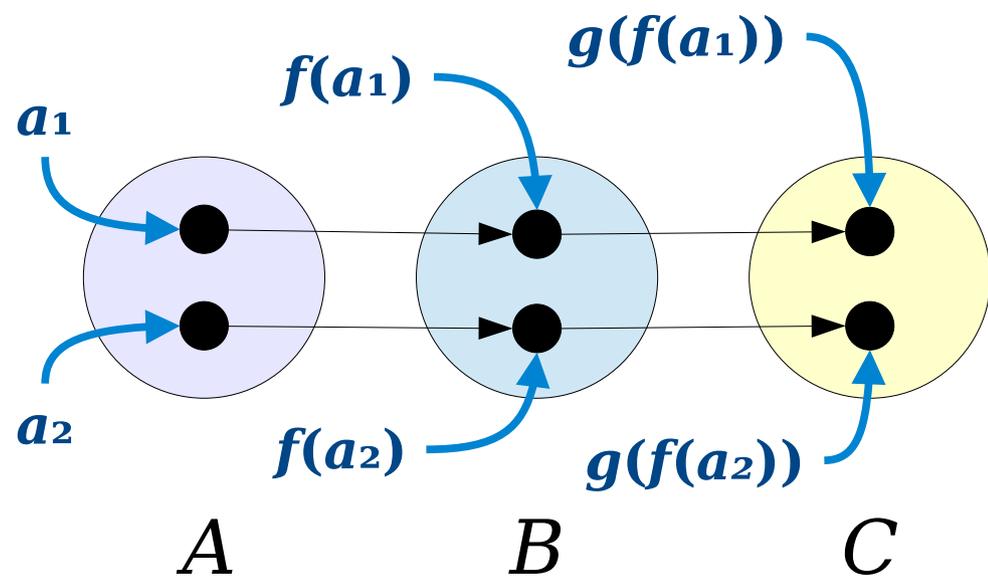
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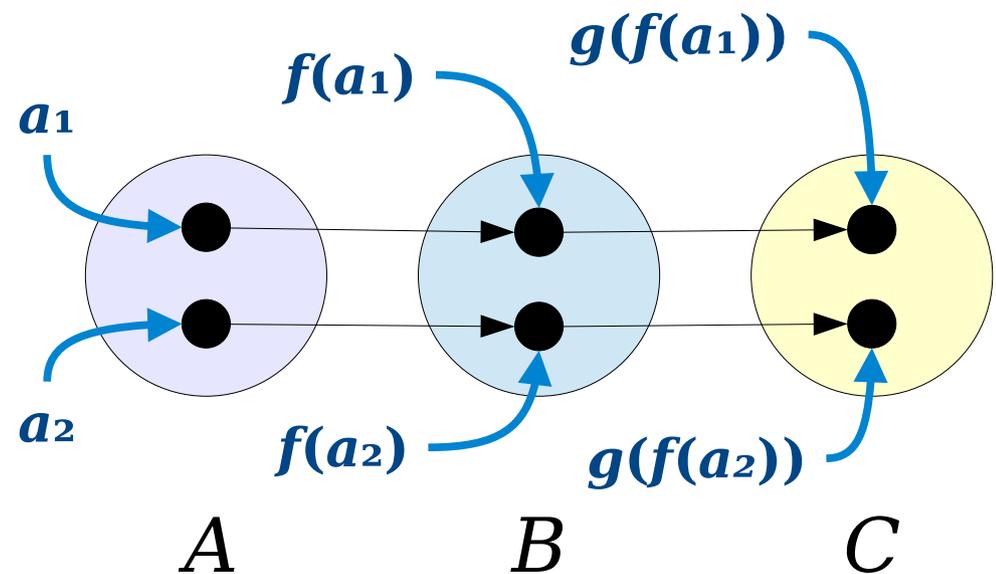


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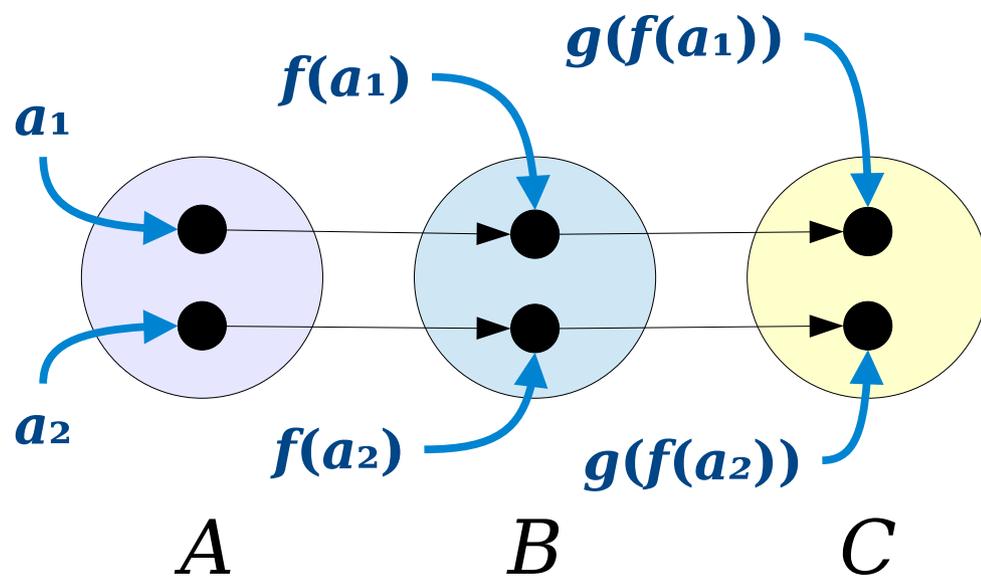
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**Proof:**



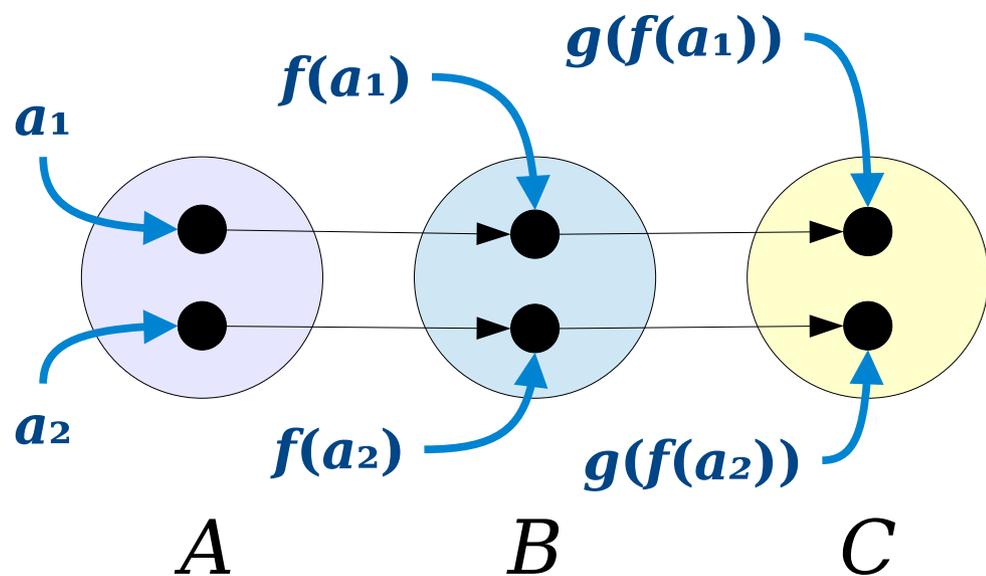
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**Proof:** Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be arbitrary injections.



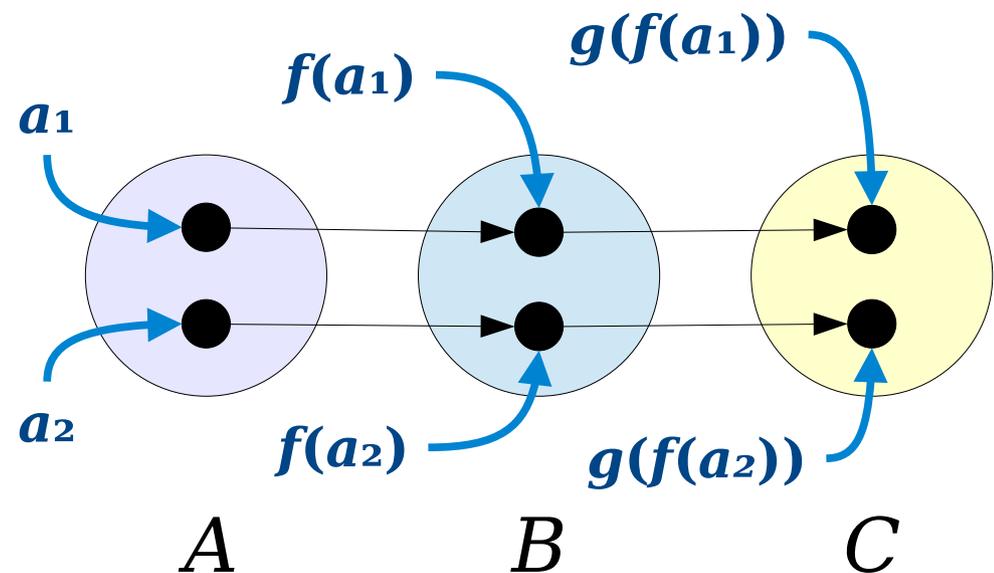
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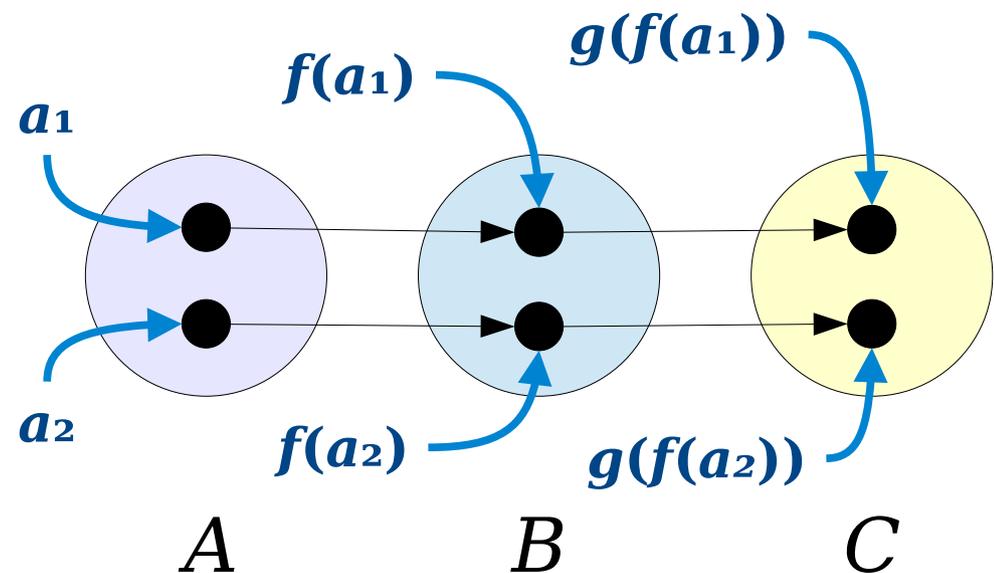
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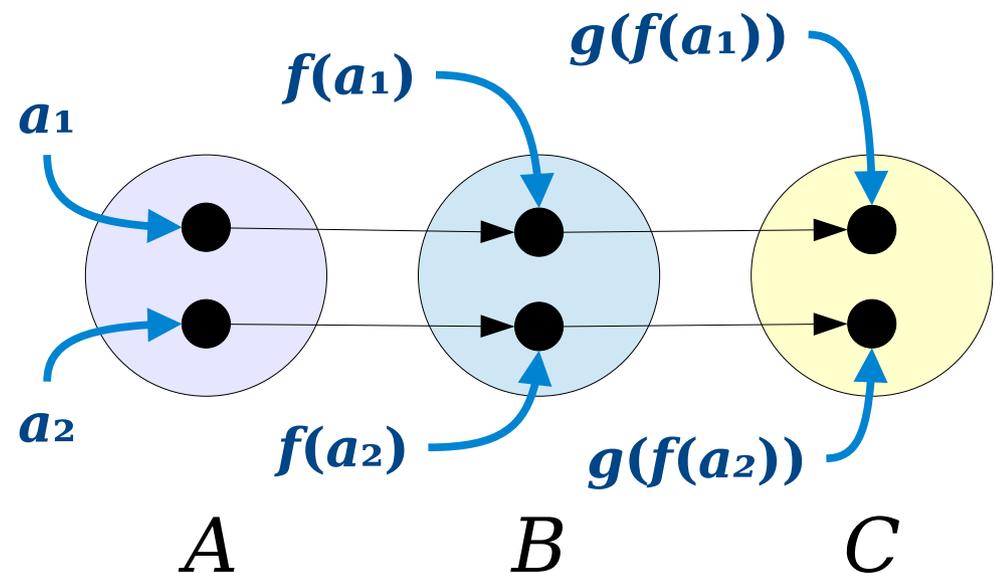
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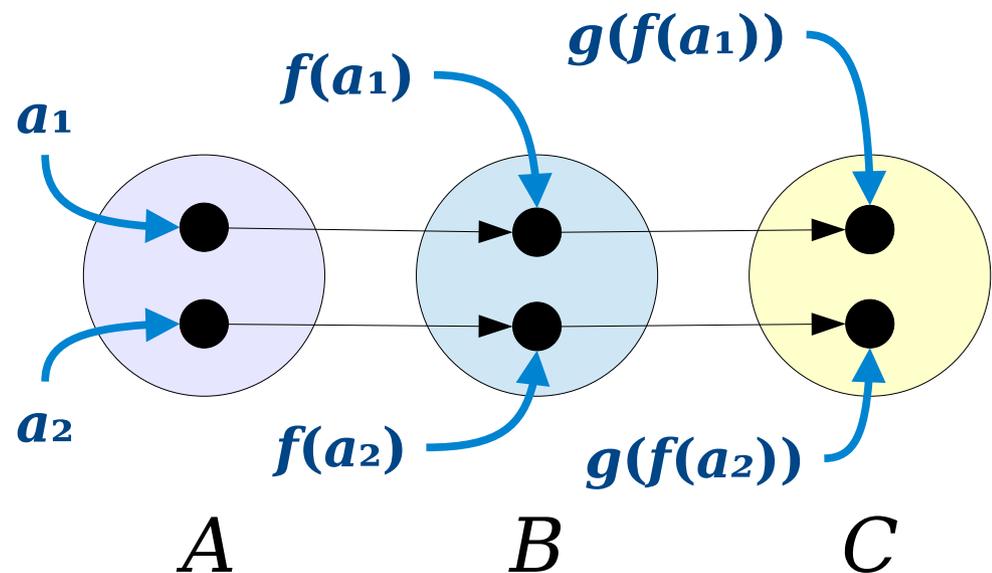
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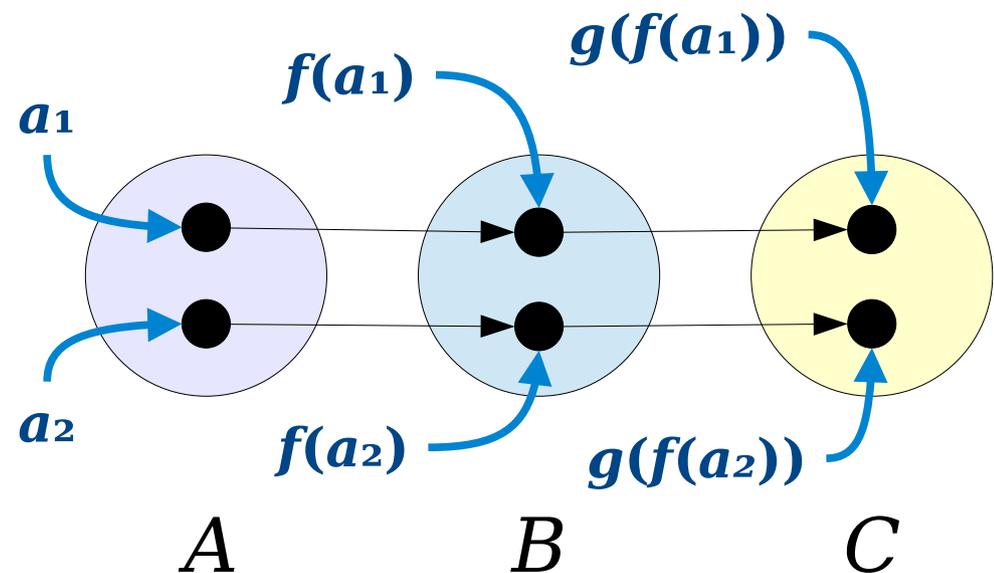
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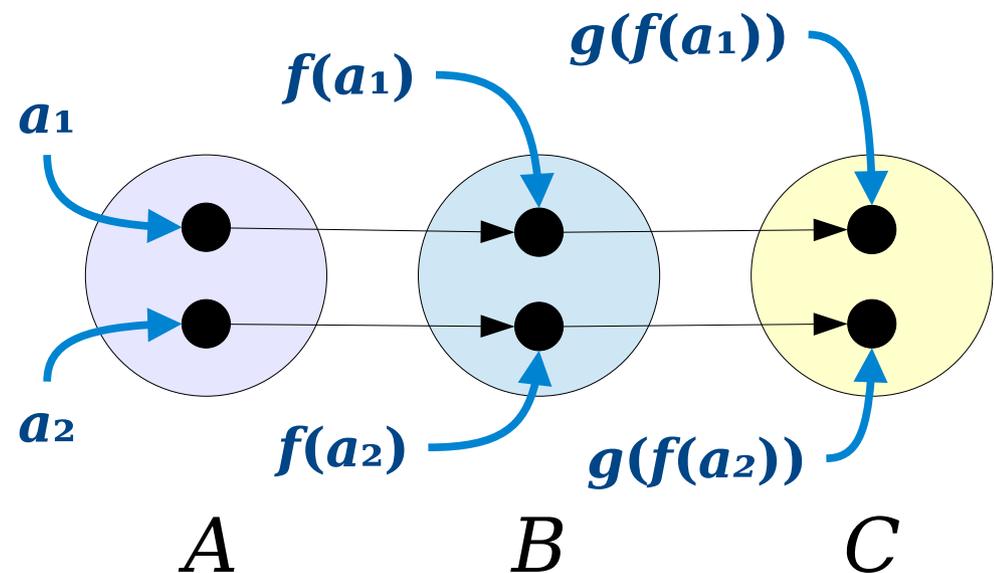
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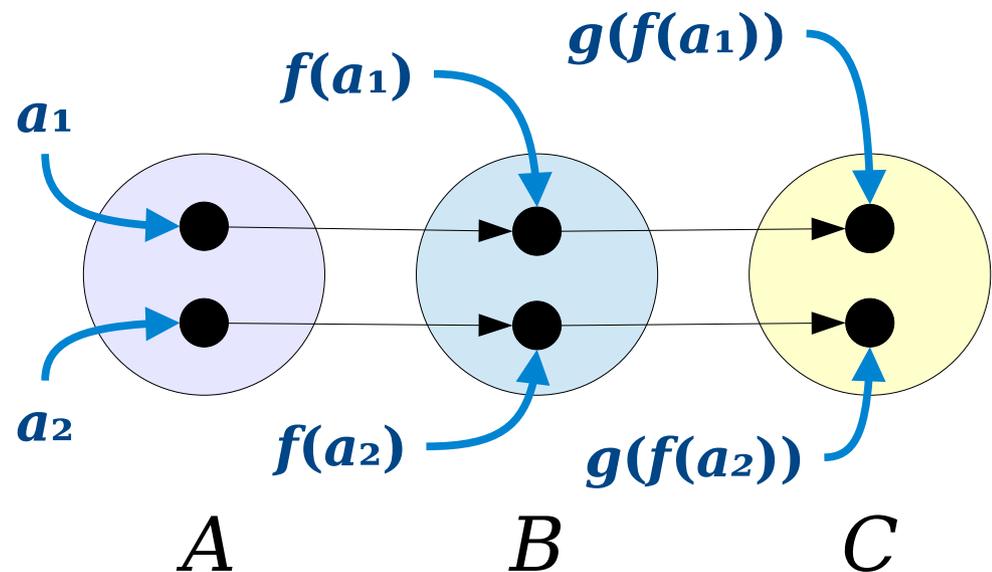


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Great exercise: Repeat this proof using the other definition of injectivity.

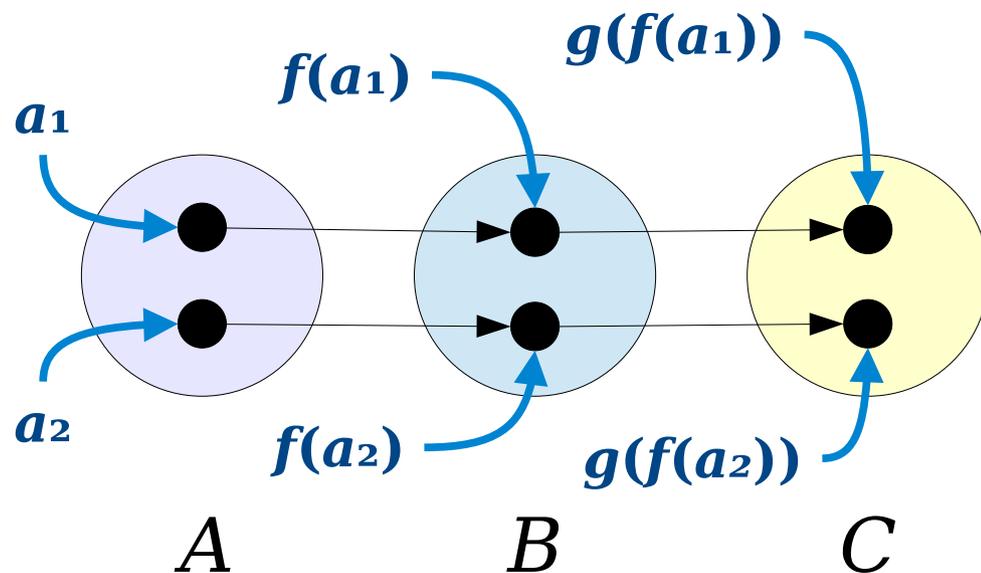


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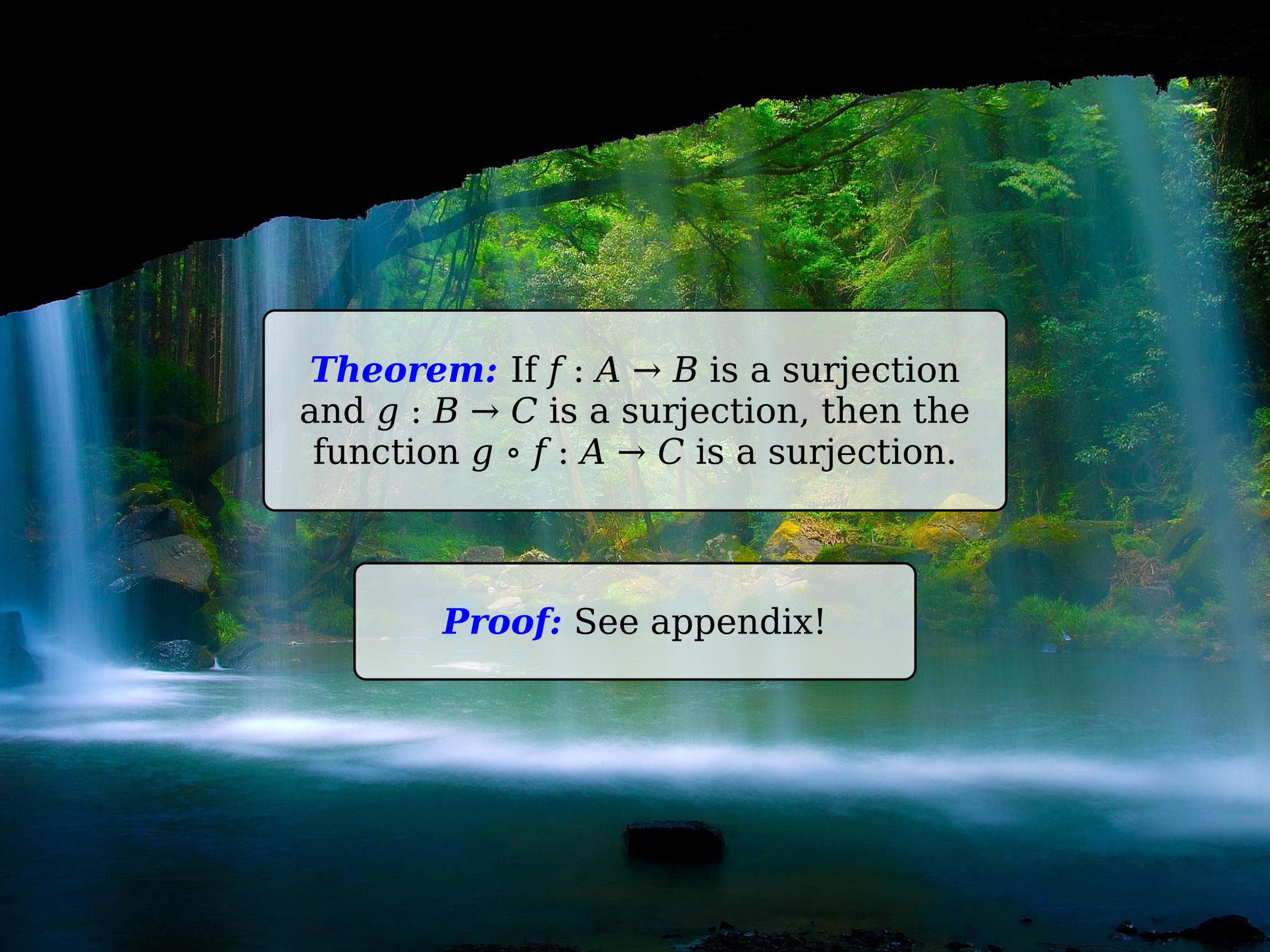
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This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.



***Theorem:*** If  $f : A \rightarrow B$  is a surjection and  $g : B \rightarrow C$  is a surjection, then the function  $g \circ f : A \rightarrow C$  is a surjection.



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**Proof:** See appendix!

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5. Connecting Function Types
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8. Function Composition
- 9. Properties of Composition: A Proof**
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# Major Ideas From Today

- Proofs involving first-order definitions are heavily based on the structure of those definitions, yet FOL notation itself does *not* appear in the proof.
- Statements behave differently based on whether you're ***assuming*** or ***proving*** them.
- When you ***assume*** a universally-quantified statement, initially, do nothing. Instead, keep an eye out for a place to apply the statement more specifically.
- When you ***prove*** a universally-quantified statement, pick an arbitrary value and try to prove it has the needed property.

# The Assume-Prove Table

	If you <i>assume</i> this is true...	To <i>prove</i> that this is true...
$\forall x. A$	Initially, <i>do nothing</i> . Once you find a $z$ through other means, you can state it has property $A$ .	Have the reader pick an arbitrary $x$ . We then prove $A$ is true for that choice of $x$ .
$\exists x. A$	Introduce a variable $x$ into your proof that has property $A$ .	Find an $x$ where $A$ is true. Then prove $A$ is true for that specific choice of $x$ .
$A \rightarrow B$	Initially, <i>do nothing</i> . Once you know $A$ is true, you can conclude $B$ is also true.	Assume $A$ is true, then prove $B$ is true.
$A \wedge B$	Assume $A$ . Also assume $B$ .	Prove $A$ . Also prove $B$ .
$A \vee B$	Consider two cases. Case 1: $A$ is true. Case 2: $B$ is true.	Either prove $\neg A \rightarrow B$ or prove $\neg B \rightarrow A$ . <i>(Why does this work?)</i>
$A \leftrightarrow B$	Assume $A \rightarrow B$ and $B \rightarrow A$ .	Prove $A \rightarrow B$ and $B \rightarrow A$ .
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

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# What's Next?

- ***Set Theory Revisited***
  - Formalizing our definitions.
- ***Proofs on Sets***
  - How to rigorously establish set-theoretic results.



**Appendix:**  
*Additional Function Proofs*

***Proof:*** Composing surjections  
yields a surjection.

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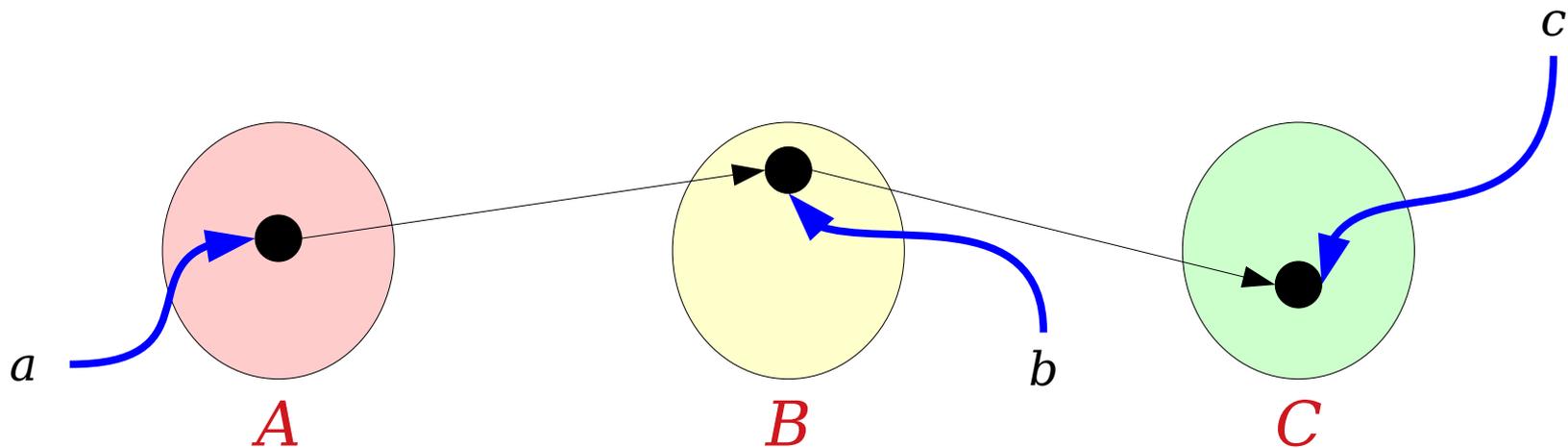
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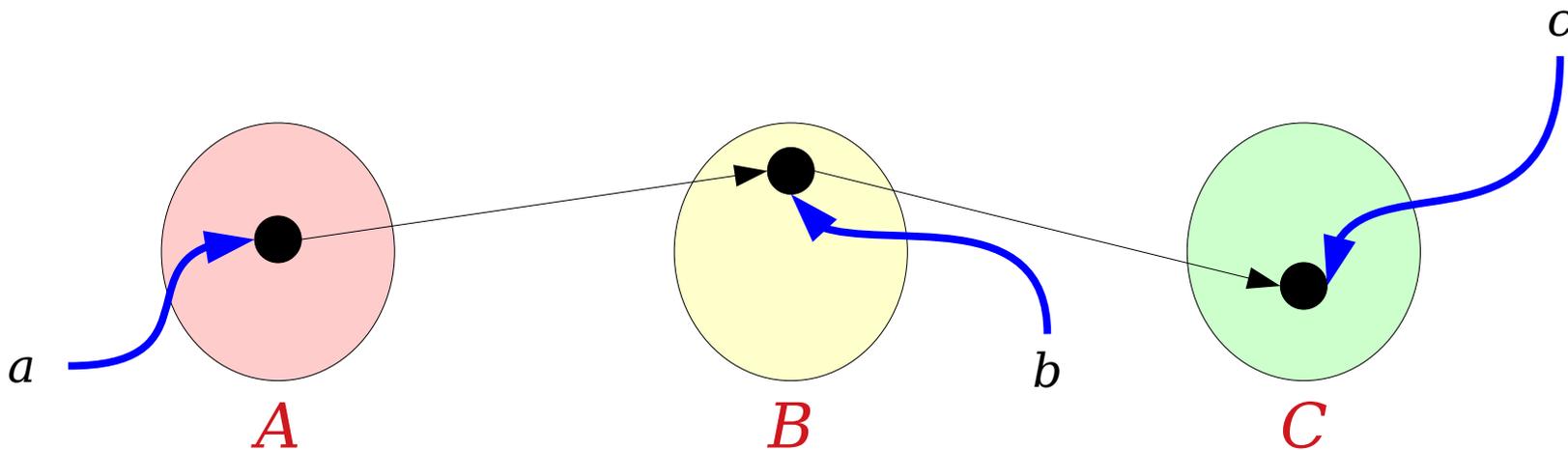
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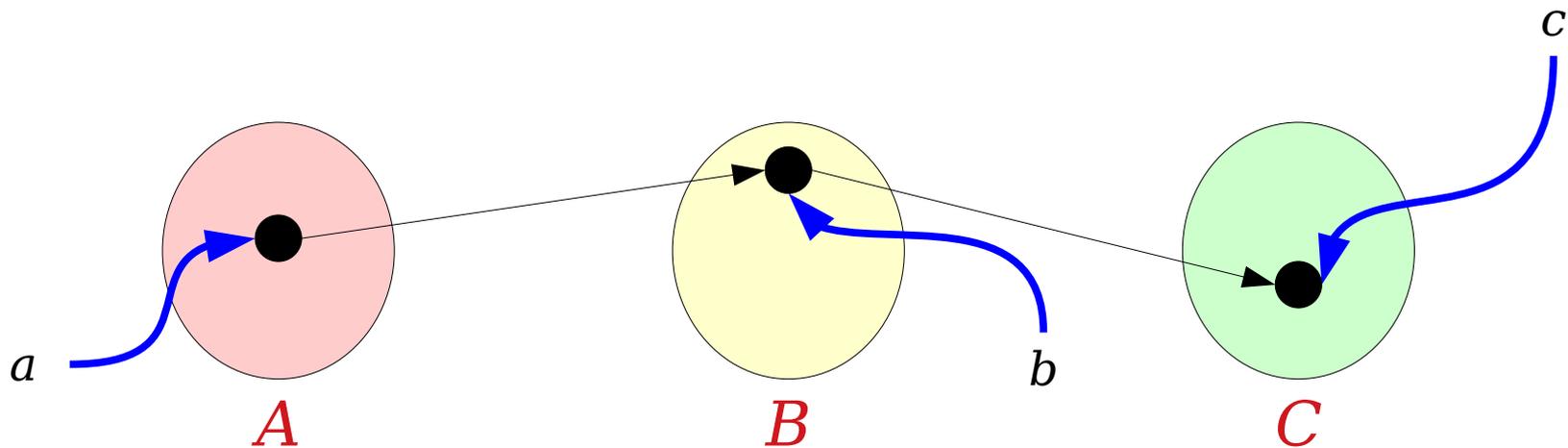
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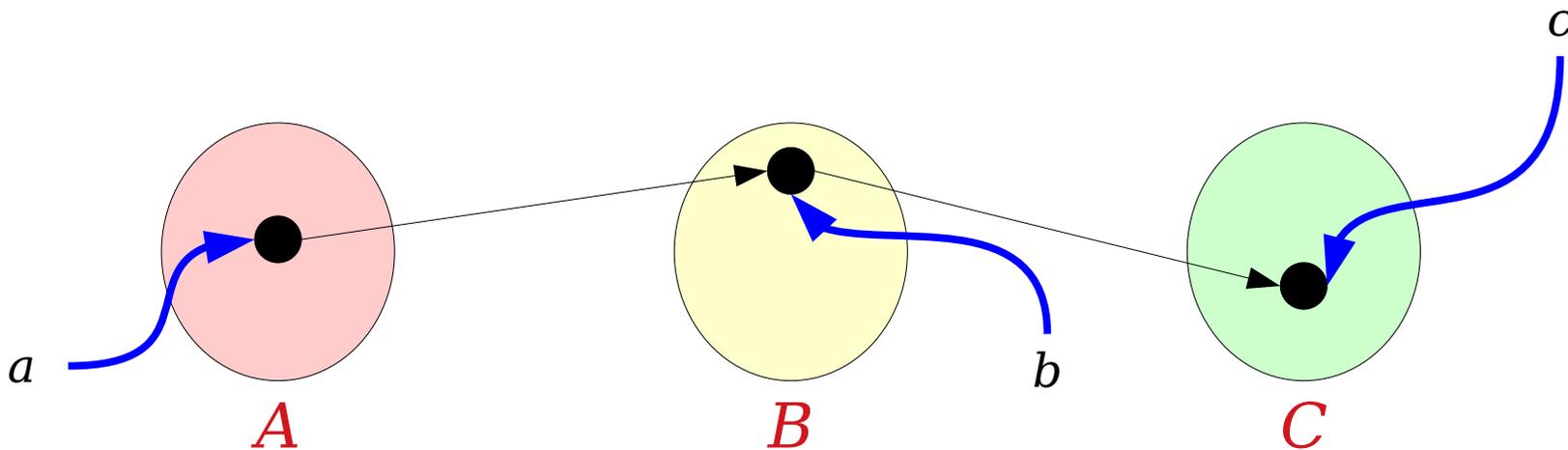
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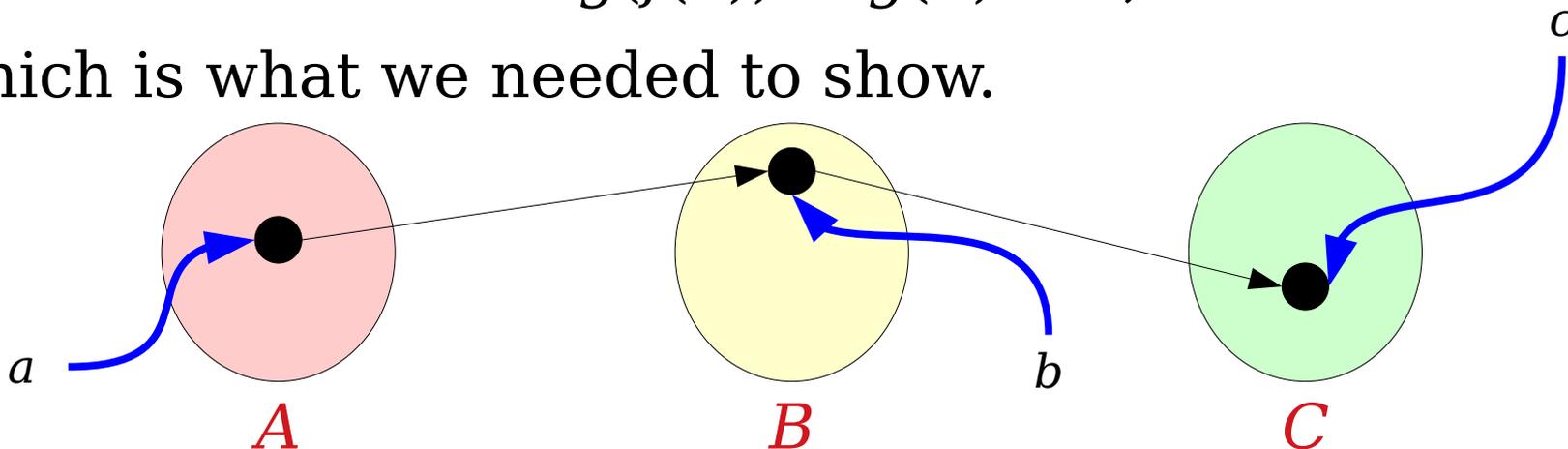
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